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Sect 1.2 (Lay's Book).

# Echelon Forms

$$A=\begin{pmatrix} 3 & 0 & -2 & -5 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad B=\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 9 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 6 & 4 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad D=\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & -3 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 9 & 4 \end{pmatrix}$$

$$E=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# 1.2 Row Reduction and Echelon Form.

Previous to the definition:

Nonzero row or Nonzero Column:

It is a row or column that contains at least one nonzero number or entry.

Leading Entry of a Row:

It is the leftmost nonzero entry in a nonzero row.

## Some important forms of matrices

Property 1:- All nonzero rows are above any row of all zeros.

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 5 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Property 2.- Each leading entry of a row is in a column to the right of the leading entry of the row above it.

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & -5 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 3 & -5 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A Consenquence of properties 1 and 2 is

Property 3.– All entries in a column below a leading entry are zeros.

Definition.

A rectangular matrix is in echelon form or row echelon form if it satisfies properties 1 and 2.

Two more important properties:

Property 4.– The leading entry in each nonzero row is 1.

Property 5.– Each leading 1 is the only nonzero entry in its column.

The previous matrices are in row echelon form but they do not satisfy properties 4 and 5.
In fact,

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 5 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 3 & -5 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

However, by performing some more row operations they can be transformed into "row-equivalent" matrices that satisfy properties 4 and 5

In fact,

$$A=\begin{pmatrix}1&2&0&0&4\\0&5&3&4&-1\\0&0&0&0&0\end{pmatrix}\overset{\frac{\text{row }2}{5}}{\sim}\begin{pmatrix}1&2&0&0&4\\0&1&\frac{3}{5}&\frac{4}{5}&\frac{-1}{5}\\0&0&0&0&0\end{pmatrix}\overset{(-2)\text{row }2+\text{row }1}{\sim}\boxed{\begin{pmatrix}1&0&-\frac{6}{5}&-\frac{8}{5}&\frac{22}{5}\\0&1&\frac{3}{5}&\frac{4}{5}&-\frac{1}{5}\\0&0&0&0&0\end{pmatrix}}$$

Also,

$$B=\begin{pmatrix}0&1&3&-5&4\\0&0&2&0&0\\0&0&0&4&5\\0&0&0&0&0\end{pmatrix}\overset{\frac{\text{row }2}{2},\ \frac{\text{row }3}{4}}{\sim}\begin{pmatrix}0&1&3&-5&4\\0&0&1&0&0\\0&0&0&1&\frac{5}{4}\\0&0&0&0&0\end{pmatrix}\sim$$

$$\overset{(-3)\text{row }2+\text{row }1}{\sim}\begin{pmatrix}0&1&0&-5&4\\0&0&1&0&0\\0&0&0&1&\frac{5}{4}\\0&0&0&0&0\end{pmatrix}\overset{(5)\text{row }3+\text{row }1}{\sim}\boxed{\begin{pmatrix}0&1&0&0&\frac{41}{4}\\0&0&1&0&0\\0&0&0&1&\frac{5}{4}\\0&0&0&0&0\end{pmatrix}}$$

<u>Definition</u> A rectangular matrix in echelon form is in <u>reduced row echelon form</u> or <u>reduced echelon form</u> if it satisfies properties 4 and 5.

## Definition

A rectangular matrix in echelon form is in Reduced Echelon form or Reduced Row Echelon form if it satisfies properties 4 and 5.

## Solutions of Linear Systems from Augmented matrix in row echelon form.

$$\begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{cases} x_3 = 1 \\ x_2 = 2 - 3x_3 = 2 - 3(1) = -1 \\ x_1 = -5 + 2x_3 = -5 + 2 = -3 \end{cases}$$

There are no free variables

"Unique Solution"

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 & -4 \end{pmatrix} \qquad \begin{cases} x_3 = -4 - 5x_4 \\ x_2 = 2 - 3x_4 \\ x_1 = -1 - 2x_4 \end{cases}$$

$x_4$ is called $\boxed{\text{a free Variable.}}$

"Infinitely many solutions"

$$\begin{pmatrix} 1 & 2 & 0 & -11 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$0 = 0$

$0 = 1$ Contradiction!

$x_3 = 5$

$x_1 = -11 - 2x_2$

"No Solutions."

Are these matrices in "Reduced Row Echelon Forms"?

Definition

Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

Theorem 1 Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

Observation: For any **echelon** form matrix of a given matrix the leading entries are always in the same position.

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A pivot column is a column of A that contains a pivot position.

From the previous observation and definition, we can also define a pivot position as a location in the original matrix A that corresponds to a leading entry in any echelon form of A.

# Section 1.2 (Lay's Book)

(Edition 4th) **Problem #3**

Leading entry 1st row

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix} \sim$$

(−2) row 1 + row 2
(−3) row 1 + row 3

$\left(-\frac{3}{2}\right)$ row 2 + row 3

first non-zero entry in column 1

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left(-\frac{1}{2}\right)$ row 2

(−4) row 2 + row 1

Pivot positions

Pivot positions

Pivot Positions

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Pivot columns. 1 and 3.

Pivot positions

Reduced Row echelon form of A

Echelon form of A

If the matrix A were the augmented matrix of a linear system then, this linear system would be

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 8 \\ 2x_1 + 4x_2 + 6x_3 &= 8 \\ 3x_1 + 6x_2 + 9x_3 &= 12 \end{aligned} \qquad A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

After the row reduction

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 + 2x_2 \quad &= -8 \\ x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

$x_1 = -8 - 2x_2$ free Variable

$x_3 = 4$

Basic Variables

They correspond to pivot positions.

General Solution:

$$\begin{cases} x_1 = -8 - 2x_2 \\ x_2 \text{ free} \\ x_3 = 4 \end{cases}$$

Infinitely many solutions. They depend on the Value of $x_2$.

If $x_2 = 1 \Rightarrow x_1 = -10, \; x_3 = 4.$

If $x_2 = -1 \Rightarrow x_1 = -6, \; x_3 = 4$

Two solutions are

$$\begin{cases} x_1 = -10 \\ x_2 = 1 \\ x_3 = 4 \end{cases}, \quad \begin{cases} x_1 = -6 \\ x_2 = -1 \\ x_3 = 4 \end{cases}$$

## THEOREM 2 Existence and Uniqueness Theorem

1. *A linear system is consistent if and only if the rightmost column of the augmented matrix is* **not** *a pivot column, that is, if and only if an echelon form of* the *augmented matrix has no row of the form* [0 . . . 0 b], *with* b ≠ 0.

2. *If a linear system is consistent, then the solution set contains either*

   (i) *a unique solution, when there are no free variables, or* Equivalent to say every column is a pivot column. of the coefficient matrix.

   (ii) *infinitely many solutions, when there is at least one free variable.*

   Equivalent to say:
   Not every column is a pivot column. of the coefficient matrix

Consider the following Coefficients matrices. in echelon form.

I) $A_1 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

\# nonzero rows = 3
Linear System will be consistent

\# free Variables = 0.
\# of Variables = 3

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II) $A_2 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$

\# nonzero rows = 3
Lin. syst. consistent
\# free Variables = 1
\# Variables = 3

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$A_3 = \begin{pmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

\# nonzero rows = 2
Linear system consistent
\# free Variables = 2
\# Variables = 4

Notice,

$$\boxed{\#\text{ free Variables} = \#\text{ Variables} - \underbrace{\#\text{ nonzero rows}}_{\substack{\text{definition} \\ \text{rank}(A)}}}$$

rank (A).

<u>Def.</u> The rank of a coefficient matrix A is the number of nonzero rows in any of their row echelon form matrices.

<u>Rank Theorem.</u>

Let A be the coefficient matrix of a system of linear equations with n variables. If the system is consistent, then

$$\boxed{\#\text{ free Variables} = n - \text{rank}(A).}$$

## Homogeneous system

$$x - y + 2z = 0$$
$$4x - 3y - z = 0$$

Constant term in each equation is zero

$\#$ equations $= m = 2$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 4 & -3 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -7 & 7 & 0 \end{array}\right)$$

$[A|0]$ free Variable

$\#$ Variables $= n = 3$, $\#$ nonzero rows in Echelon form matrix $=$ rank (coeff. matrix) $= 2$

$\#$ free Variables $= n - \text{rank}(A) = 3 - 2 = 1$.

**Thm 2.3** If $[A|0]$ corresponds to a homogeneous system of $m$ equations with $n$ Variables and $m < n$, then the system has infinitely many solutions.

Proof:–

i) $\#$ nonzero rows $\le m < n \rightarrow \text{rank}(A) < n$

ii) Also, the system is Consisting

then, according to rank theorem

$\#$ free Variables $= n - \text{rank}(A) > 0 \longrightarrow$ Infinitely many Solutions.

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## 2.2 # 30) Gauss-Jordan Elimination

$$-x_1 + 3x_2 - 2x_3 + 4x_4 = 0$$
$$2x_1 - 6x_2 + x_3 - 2x_4 = -3$$
$$x_1 - 3x_2 + 4x_3 - 8x_4 = 2$$

$$\begin{pmatrix} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{pmatrix}$$

Nrow 1 = (−1) * row 1

Nrow 2 = (−2) * Nrow 1 + row 2

Nrow 3 = (−1) * Nrow 1 + row 3

leading entry　free Variables

$$\sim \begin{pmatrix} \boxed{1} & -3 & 2 & -4 & 0 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Nrow 2 = $-\frac{1}{3}$ * row 2

Nrow 3 = (−2) Nrow 2 + row 3

In row echelon form

$$\begin{array}{ccccc} \text{LE} & \text{FV} & \text{LE} & & \\ \end{array}$$

$$\sim \begin{pmatrix} \boxed{1} & -3 & 0 & 0 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FV

Nrow 1 = (−2) row 2 + row 1

In reduced row echelon form

Backward Substitution

$$x_3 = 1 + 2x_4$$ Free Variables

$$x_1 = -2 + 3x_2$$

So Vector Solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 + 3x_2 \\ x_2 \\ 1 + 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

2.2.

#46) Line of intersection of planes:

$4x + y + z = 0$ and $2x - y + 3z = 2$.

$$\left(\begin{array}{ccc|c} 4 & 1 & 1 & 0 \\ 2 & -1 & 3 & 2 \end{array}\right) \sim \begin{pmatrix} 2 & -1 & 3 & 2 \\ 4 & 1 & 1 & 0 \end{pmatrix}^{(-2)}$$

$$\sim \begin{pmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -5 & -4 \end{pmatrix}$$

$3y = -4 + 5z$ (1)

$2x = 2 + x_2 - 3z$ (2)

or

or $\boxed{y = -\frac{4}{3} + \frac{5}{3}z}$

$2x_1 = 2 - \frac{4}{3} + \frac{5}{3}z - 3z$

$2x = \frac{2}{3} - \frac{4}{3}z$

$\boxed{x = \frac{2}{6} - \frac{4}{6}z}$

Then,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{4}{6}z + \frac{2}{6} \\ \frac{5}{3}z - \frac{4}{3} \\ z \end{bmatrix} = z \begin{bmatrix} -\frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 0 \end{bmatrix}$$

So line of intersection:

$\boxed{\vec{x} = t\vec{v} + \vec{p}}$, $t \in \mathbb{R}$, $\vec{v} = \begin{pmatrix} -2/3 \\ 5/3 \\ 1 \end{pmatrix}$

$\vec{p} = \begin{pmatrix} 1/3 \\ -4/3 \\ 0 \end{pmatrix}$